FROM DISCRETE TO CONTINUOUS SYSTEMS. NON-LOCALITY AND FRACTIONAL CALCULUS.

LUIS VÁZQUEZ[1], M. PILAR VELASCO[2], SALVADOR JIMÉNEZ[3],

Abstract. We present a panoramic view of the relation between continuous and discrete systems as well as one of the transitions for local to nonlocal differential equations.

Keywords: Non-locality, Fractional Differential Equations, Discrete Systems.

1. Introduction

We present a short panoramic view of one of the lines of work in our group at Complutense University of Madrid. This research line is related to the emergent lines of research and technological applications. This involves the following conceptual issues:

- Discrete (Continuous) Systems versus Continuous (Discrete) Systems. The measure of the difference between the dynamics of the two limits is a challenging issue. How close are they? How suitable are the related approximations?.
- The non-locality either in space and/or time represents a modeling instrument to understand in a deeper way how is the dynamics of the system. This context allows to implement the Fractional Calculus in the framework of nonlocal systems in a natural way.

Given a system modeled by continuous differential equations, usually we have to approach numerically the solutions. This is carried out by a suitable numerical scheme which is essentially discrete. On the other hand, given an essential discrete system, we can obtain useful information by analyzing several kinds of continuous limits.

The nonlocal character makes the fractional derivatives suitable for the modellization of systems with long-range interactions either in space or depending of the history of the systems. On the other hand, the freedom in the definition of fractional derivatives allows us to incorporate different types of information. At the same time, the fractional derivatives with noninteger exponents stress which algebraic scale properties are relevant to the data analysis.

This paper is organized as follows. In Section 2, we show a panoramic view of the non-locality through important equations of modellization in the literature and we remark that these classical equations provide the basis to introduce the fractional differential equations. A more detailed study of the different fractional context is presented in Section 3; here we show many applied scenarios where the fractional operators appear of natural form.

2. From non-locality to fractional differential equations

The study of nonlocal wave equations is an introduction to the study of fractional differential equations. The fractional derivatives model the nonlocal effects in a natural way.

One of the first nonlocal equations in the modern literature appears in the study of waves in fluids. T.J. Benjamin (1967) [8] and H. Ono (1975) [34] propose a similar equation to the equation of Kroteweg and de Vries for the study of waves in fluids,

$$\frac{\partial u}{\partial t} + \frac{c\delta}{2} H \frac{\partial^2 u}{\partial x^2} + \frac{9c}{2} u \frac{\partial u}{\partial x} = 0, \tag{1}$$

where the operator H denotes the Hilbert transform

$$Hu(x) = \frac{1}{\pi} v.p. \int_{-\infty}^{\infty} \frac{u(x')}{x' - x} dx'. \tag{2}$$

The consequence of the inclusion of the integral transform is that the solitary waves take an algebraic asymptotic behavior slower than the exponential one.

On the same date, G. Whitham [48] proposes an equation with a generalized kernel,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \int_{-\infty}^{\infty} k(x-s) \frac{\partial u}{\partial s} ds = 0, \tag{3}$$

that is known as Whitham equation or Whitham-Benjamin equation.

The motivation for the study of this equation is to find a description of phenomena of the waves as, for example, the formation and the breakdown of sharp crests, or the crashing waves. These phenomena was not explained through the classical equations.

Later, this equation has been studied by including other type of kernels, as the exponential kernels with parametric dependence of Eleonsky et al. [16].

The Whithman equation has been generalized to the expression

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \mathbb{K}(u) = 0, \tag{4}$$

where the operator $\mathbb{K}(u)$ is defined as

$$\mathbb{K}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{tpx} K(p) \hat{u}(p,t) dp, \tag{5}$$

and $\hat{u}(p,t)$ is the Fourier transform of $u(x,t)$. The function $k(p)$, that defines the nonlocal operator $\mathbb{K}(u)$, is known as symbol of the operator. The election of the symbol function can generate many equations studied in hydrodynamics. So, $k(p) = i\xi p^3$ recovers the equation of Korteweg and Vries and $k(p) = vp^2$ recovers the Burgers equation,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - v \frac{\partial^2 u}{\partial x^2} = 0. \tag{6}$$

Other problems allow a formulation similar to the Whithman equation with different kernels. Waves in different fluids, problems to acoustic pressure, wave transmission under the ice, modelization of flame fronts, bidimensional turbulences, capillarity waves... A detailed description of nonlocal problems in hydrodynamics and its mathematical study can be read in [33].

However, the hydrodynamic is only one field where the integrodifferential equations appear. So far, the integral in the equations represents a dispersion term, such that the inclusion of different kernels produces an arbitrary dispersion in the waves. Similar terms appear in others scientific fields.

For example, Y. Ishimori [26] studies a one dimensional chain of atoms with Lennard-Jones interaction $(2n, n)$, where n is the potential exponent. In this work, he proves that the system can be described through a Korteweg-de Vries equation in the continuous approximation for $n \geq 4$ and then the system presents the characteristics of this equation. On the other hand, for $n = 2$ the equation is a Benjamin-Ono equation (1). For $n = 3$ the equation is nonlocal too.

Although the study of Ishimori does not introduce a significative nonlocal nonlinear equation, this work is the beginning of the study of a fruitful field to nonlocal nonlinear equations. The study of long-range interactions among atoms or molecules of a crystalline net was a work to avoid and interactions among close neighbors are considered. In this form, the system is more simple and local partial derivatives equations are obtained in general.

The introduction of long-range interparticle interactions produces similar equations to the equations of local models but the dispersive term ($\partial^2 u/\partial x^2$ or similar others) is replaced by an integral term.

One of these equations is a generalization of the classical sine-Gordon equation, that is known as nonlocal sine-Gordon equation,

$$\frac{\partial^2 u}{\partial t^2} + \sin(u) - \frac{\partial}{\partial x}\int_{-\infty}^{\infty} G(x-x')\frac{\partial u}{\partial x'}dx', \tag{7}$$

where $G(\rho)$ is the kernel of the interaction. This equation appears in two fields: the crystalline solids and the study of superconductivity.

In the line of Ishimori, Pokrovsky et al. [37] study the effect of the long-range interaction in the phase transitions and they obtain the equation (7) in the approximation to the continuous. Remoissen et al. [38] study the effect of long-range interactions of exponential type in the solitons of one dimensional anharmonic chains and they obtain a similar equation. Other models where the equation (7) appear in the approximation to the continuous are the model of Frenkel-Kontorova with long-range interaction $1/x^n$ [9] or the sine-Gordon systems with an interparticles interaction described by a potential of Kac-Baker [49]. Also, a similar equation appears in other fields of Physics.

The sine-Gordon equation appear in the study of superconductivity phenomena in Josephson junctions when we suppose that the sheets of supercurrent material are relatively heavy and the penetration depth of the magnetic field or the London length is small in comparison to the Josephson length that determines the size of the structures in the junction. However these conditions are not always verified and then the obtained equations are as (7) with different kernels $G(\rho)$. This is the case of Ivachenko et al. [27], that study the case when the thickness of the sheets is of the same order that the London length and they obtain a similar equation. Gurevich [24], [25] study the case in which the Josephson length is of the order of the London length.

More general cases have been study by Vázquez et al. [44], [12], [18], and Alfimov et al. [1], [2], [3], [6]. These works present a numerical and theoretical analysis of different kernels to the previous equation that allows to extend the solutions to more general cases. In the last work, the authors extend an exponential kernel of Kac-Baker to a more generic kernel that can be expressed as a Laplace transform.

Also, Alfimov et al. [4], [5] study an equation known as sine-Hilbert-II equation,

$$\frac{\partial^2 u}{\partial t^2} + \sin(u) = H\left[\frac{\partial u}{\partial x}\right], \tag{8}$$

where $H[u]$ denotes the Hilbert transform (2). This equation is a specific case of the equation (7).

In the previous cases, the nonlocal equation presents the same solutions that the local equation (periodic, rotational and kink). Also, in some cases, the equation presents multikink solutions, that is kink solutions that joint non-consecutive empty states. This is not possible in the classical sine-Gordon equation. Also, the existence of special solutions, the breathers, is proved; they are solutions located in the space and periodic in the time, and the local equation presents these solutions too.

The nonlocal version of the Schrodinger equation is other example of nonlocal nonlinear equation that appears as a generalization of a nonlinear equation.

There exist many different generalizations that produce very different equations. In the previous line, a generalizations is the substitution of the dispersive term by an integral term. In this sense, Gaididei et al. [19], [21], [20], [22] study lattices of particles that interact according to close neighbors (local model), an interaction $1/r^2$, or an interaction Kac-Baker (decreasing exponential). The obtained equation in the approximation to the continuous is

$$i\frac{\partial u}{\partial t} + \int_{-\infty}^{\infty} G(x-x')u(x')dx' + |u|^2 u = 0, \tag{9}$$

where the kernel $G(\rho)$ can contain a derivative operator. The properties of the solutions change respect to the solutions of the local equation, by producing phenomena as, for example, the existence of different solutions with the same norm or theirs algebraic localization.

In all these cases, there exist applied examples where the derivative operator is generalized to a more general integrodifferential operator. This is the case of the fractional derivatives, that generalize the concept of derivative and that sometimes include operators as the operators of equations (3), (4) or (6).

There exist many definitions of fractional derivatives and integrals. The more used definitions in applications are the following operators.

Definition 1. (*Riemann-Liouville fractional integrals*)*: Let* $\alpha \in \mathbb{C}$*, with* $\Re(alpha) > 0$ *and* $n = [\Re(\alpha)] + 1$ $(n \in \mathbb{N})$*,* $[a,b] \subset \mathbb{R}$ *and let* f *a suitable real function real* (*for example, it suffices* $f \in L_1(a,b)$)*. Riemann-Liouville fractional integrals are defined as:*

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\ dt \qquad (x>a) \tag{10}$$

$$(I^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\ dt \qquad (x<b) \tag{11}$$

As inverse operation to fractional integral, the Riemann-Liouville fractional derivatives are obtained on finite interval:

Definition 2. (*Riemann-Liouville fractional derivatives*)*: Let* $\alpha \in \mathbb{C}$*, with* $\Re(alpha) > 0$ *and* $n = [\Re(\alpha)]+1$ $(n \in \mathbb{N})$*,* $[a,b] \subset \mathbb{R}$ *and let* f *a suitable real function* (*for example, it suffices* $f \in L_1(a,b)$)*. Riemann-Liouville fractional derivatives are defined* *as:*

$$(D^{\alpha}_{a+}f)(x) = D^n(I^{n-\alpha}_{a+}f)(x) \qquad (x > a) \tag{12}$$
$$(D^{\alpha}_{b-}f)(x) = (-D)^n(I^{n-\alpha}_{b-}f)(x), \qquad (x < b) \tag{13}$$

where D is the ordinary differential operator.

These operators recover the classical operators for the parameter $\alpha = 1$ and the algebra of these operators is different to the classical operators:

Property 1. *:* Let $f \in L_p(a,b)$ $(1 \le p \le \infty)$ and $\Re(\alpha), \Re(\beta) > 0$. Then:

$$(I^{\alpha}_{a+}I^{\beta}_{a+}f)(x) = (I^{\alpha+\beta}_{a+}f)(x) \tag{14}$$

in $[a,b]$.

Property 2. *:* Let $f \in L_1(a,b)$, $\alpha, \beta > 0$, such that $n-1 < \alpha \le n$, $m-1 < \beta \le m$ $(n, m \in \mathbb{N})$ and $\alpha + \beta < n$, $f_{m-\alpha} = I^{m-\alpha}_{a+}f \in AC^m([a,b])$. Then:

$$(D^{\alpha}_{a+}D^{\beta}_{a+}f)(x) = (D^{\alpha+\beta}f)(x) - \sum_{j=1}^{m}(D^{\beta-j}_{a+}f)(a+)\frac{(x-a)^{-j-\alpha}}{\Gamma(1-j-\alpha)}. \tag{15}$$

Other important definition of fractional derivative that is very used in modeling is the Caputo fractional derivative:

Definition 3. (*Caputo fractional derivative*)*: Let $\alpha \in \mathbb{C}$, with $\Re(alpha) > 0$ and $n = [\Re(\alpha)] + 1$ $(n \in \mathbb{N})$, $[a,b] \subset \mathbb{R}$ and let f a suitable real function* (*for example, it suffices $f \in L_1(a,b)$*)*. Caputo fractional derivative is defined as:*

$$({}^C D^{\alpha}_{a+}f)(x) = (I^{n-\alpha}_{a+}D^n f)(x) \qquad (x > a). \tag{16}$$

The following identity established the relation between the Riemann-Liouville and Caputo fractional derivatives, for f a suitable function (for example, f n-derivable):

$$(D^{\alpha}_{a+}f)(x) = ({}^C D^{\alpha}_{a+}f)(x) + \sum_{j=0}^{n-1}\frac{f^{(j)}(a)}{\Gamma(1+j-\alpha)}(x-a)^{j-\alpha}. \tag{17}$$

Fractional operators are nonlocal and they keep memory of the previous states. In this sense, fractional operators can be applied in long-range interactions systems.

3. From discrete classical to fractional differential equations

Following the models of classical mechanics [23] and quantum mechanics [15], let us consider a possible standard path among the basic equations of physics that would allow us to interpret, in a more wide context, the level of fractionalization of the basic differential equations by analyzing the associated dynamics and solutions in the framework of the corresponding modellization [32], [29], [41], [28], [31].

For the sake of simplicity, let us consider systems in one space dimension. Newtons equation for a particle of mass m moving in a one-dimensional force field F, u being the displacement, is

$$m\frac{d^2u}{dt^2} = F. \tag{18}$$

Let us consider now a discrete system, formed by infinite point-like masses, m, spaced a distance L, and connected by strings of the same constant k. If we name

u_i the displacement of the particle i from the equilibrium, the equation of motion is given by

$$\frac{m}{L}\frac{d^2u_i}{dt^2} = kL\frac{u_{i+1} - 2u_i + u_{i-1}}{L^2}. \tag{19}$$

This equation is written in such a way as to interpret appropriately the continuous limit of the system when $L \to 0$:

$$\begin{gathered} u_i \to u(x), \frac{m}{L} \to \rho \text{ (linear mass density)} \\ kL \to Y \text{ (Young's modulus)} \\ \frac{u_{i+1} - 2u_i + u_{i-1}}{L^2} \to \frac{d^2u}{dx^2}. \end{gathered} \tag{20}$$

Thus, we have the equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{21}$$

where $c^2 = Y/\rho$. With this mechanical approach, we get the wave equation which appears in many contexts according to the different meanings of u and c. If we assume a damping mechanism either in the discrete system or in the continuous limit, we have the equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{D}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{22}$$

When the damping term, $\frac{1}{D}\frac{\partial u}{\partial t}$, dominates over the inertial one, $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$, we have the standard diffusion equation

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0. \tag{23}$$

Other basic equations are obtained by considering extra linear terms. This is the case of the Klein-Gordon equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \mu^2 u = 0, \tag{24}$$

and the Telegraph equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{D}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \mu^2 u = 0. \tag{25}$$

All this can be summarized in the following equation, where we also include Dirac's equation to be discussed in the following paragraphs.

Newton's equation for one particle $\rightarrow$ Newton's equation for one-dimensional system of particles linearly coupled

System of particles linearly coupled $\overset{continuouslimit}{\longrightarrow}$ One-dimensional wave

Wave equation with damping $\rightarrow$ Diffusion equation $\rightarrow$ Fractional Dirac equation

Klein-Gordon equation $\rightarrow$ Dirac equation, Telegraph equation

Fractional calculus [40] offers a very suggestive and stimulating scenario where we have the convergence of deep and fundamental mathematical questions, development of appropriate numerical algorithms, as well as the applications to modelizations in different frameworks. Thus Fractional Calculus has many applications in different areas, as it is cited in Magin [30]:

"The purpose of this book is to explore the behavior of biological systems from the perspective of fractional calculus. Fractional calculus, integration and differentiation

of an arbitrary or fractional order, provides new tools that expand the descriptive power of calculus beyond the familiar integer-order concepts of rates of change and area under a curve."

"Fractional Calculus adds new functional relationships and new functions to the familiar family of exponentials and sinusoids that arise in the realm of ordinary linear differential equations."

Fractional operators are a powerful mathematical tool that establish important relations between transform integrals and special functions, [13], [14], [10]. Also, fractals and Fractional Calculus create intermediate-order parameters: dimensions, integration and derivative of arbitrary order. This has been studied in the literature broadly ([39], [47]), and it has allowed to get a better modeling in different applications.

From a mathematical point of view, the modelization of the long-range dependence and systems with memory are associated with integrodifferential equations in a broad sense. On the other hand, in many cases such integrodifferential equations can be understood as fractional differential equations, and they can be studied in the fractional calculus framework.

For example, we can consider different contexts of the Classical Physics, where the equations are supported by similar laws:

- Hooke's Law: $F(t) = kx(t)$
- Newton's Fluid Law: $F(t) = k\dfrac{dx}{dt}(t)$
- Newton's Second Law: $F(t) = k\dfrac{d^2x}{dt^2}(t)$
- Other possible fractional context: $F(t) = k\dfrac{d^\alpha x}{dt^\alpha}(t)$

Also time-fractional derivative have been introduced to interpolate the diffusion and wave equations [17] and to model the anomalous diffusion [35, 36, 42, 45]. In this sense, there exist many contexts where the diffusion process appears associated to the same basic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{26}$$

as it is shown in Table 1.

Law	Darcy: $\overrightarrow{q} = -K\, \overrightarrow{Grad}\, h$	Fourier: $\overrightarrow{Q} = -\kappa\, \overrightarrow{Grad}\, T$	Fick: $\overrightarrow{f} = -D\, \overrightarrow{Grad}\, C$	Ohm: $\overrightarrow{j} = -\sigma\, \overrightarrow{Grad}\, V$
Flux	Subterranean Water: q	Heat: Q	Solute: f	Charge: j
Potential	Hydrostatic Charge: h	Temperature: T	Concentration: C	Voltage: V
Medium's property	Hydraulic Conductivity: K	Thermal Conductivity: κ	Diffusion Coefficient: D	Electrical Conductivity: σ

TABLE 1. Diffusion processes

The above equation can be generalized through fractional operators and this allows to obtain a natural interpolation between equations:

$$\text{Diffusion Equation (parabolic):}\ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{27}$$

$$\text{Interpolation:}\ \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} \tag{28}$$

$$\text{Waves Equation (hiperbolic):}\ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{29}$$

Associated to this, other fractional context is the use of Dirac fractional Equations, following this scheme:

$$\begin{tikzcd}
A\dfrac{\partial\psi}{\partial t} + B\dfrac{\partial\psi}{\partial x} = 0 \arrow[r, "A\frac{\partial^\alpha\psi}{\partial t^\alpha} + B\frac{\partial\psi}{\partial x} = 0", "\psi = \begin{pmatrix}\varphi \\ \xi\end{pmatrix}"'] & A\dfrac{\partial^{1/2}\psi}{\partial t^{1/2}} + B\dfrac{\partial\psi}{\partial x} = 0 \\
\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0 \arrow[u, "\begin{matrix} A^2 = I \\ B^2 = I \\ \{A, B\} = 0 \end{matrix}"] \arrow[r, "\gamma = 2\alpha", "\frac{\partial^\gamma u}{\partial t^\gamma} - \frac{\partial^2 u}{\partial x^2} = 0"'] & \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0 \arrow[u]
\end{tikzcd}$$

Then, the equation

$$A\frac{\partial^{1/2}\psi}{\partial t^{1/2}} + B\frac{\partial\psi}{\partial x} = 0 \tag{30}$$

can be explained as the description of two coupled diffusion processes or one diffusion process with two internal degrees of freedom. In this equation, both component φ and ξ satisfy the standard diffusion equation and they are referred to as *difunors* analogous to the *spinors* of Quantum Mechanic. This provides other form to study the interpolation between the hyperbolic operator of the waves equation and the parabolic operator of the classical diffusion. By following the representation of Pauli's Algebra for A and B, we have a system of coupled or no-coupled equations

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{cases} \partial_t^\alpha \varphi = \varphi \\ \partial_t^\alpha \xi = -\xi \end{cases} \tag{31}$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{cases} \partial_t^\alpha \varphi = -\varphi \\ \partial_t^\alpha \xi = -\xi \end{cases} \tag{32}$$

$$A\frac{\partial^\alpha\psi}{\partial t^\alpha} + B\frac{\partial\psi}{\partial x} = 0 \xrightarrow{\gamma = 2\alpha} \frac{\partial^\gamma u}{\partial t^\gamma} - \frac{\partial^2 u}{\partial x^2} = 0$$

From the study of the time inversion ($t \to -t$) we have:

- For $\alpha = 1$, Dirac and waves equations are invariant by time inversion.
- For $\alpha = 1/2$, the classical diffusion equation and its square root are not invariant by time inversion.

- Interpolation for: $0 < \alpha < 1$. The invariance by time inversion is satisfied by
  - Dirac fractional equation:
    $\alpha = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, ..., \frac{3}{5}, \frac{3}{7}, \frac{3}{9}, ..., \frac{5}{7}, \frac{5}{9}, \frac{5}{11}, ...$
  - Diffusion fractional equation:
    $\alpha = \frac{1}{3}, \frac{2}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{7}, \frac{2}{7}, ..., \frac{6}{7}, \frac{1}{9}, ...$

From the study of the space-time inversion ($x \to -x$, $t \to -t$) we observe that both equations are invariant by space inversion and for the interpolation $0 < \alpha < 1$ the invariance by space-time inversion is satisfied for the same values of α in both equations:

$$\alpha = \frac{1}{3}, \frac{2}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{7}, \frac{2}{7}, ..., \frac{6}{7}, \frac{1}{9}, ...$$

Dirac fractional equation is not invariant by time translation because of the non-local character of the time fractional derivative.

Other fractional differential equations are obtained by considering the cube root of the waves and diffusion equations:

$$\text{Waves Equation: } P\partial_t^{2/3}\varphi + Q\partial_x^{2/3}\varphi = 0 \tag{33}$$

$$\text{Diffusion Equation: } P\partial_t^{1/3}\varphi + Q\partial_x^{2/3}\varphi = 0 \tag{34}$$

where

$$P^3 = I \qquad Q^3 = -I \qquad PPQ + PQP + QPP = 0 \qquad QQP + QPQ + PQQ = 0 \tag{35}$$

A possible development is in terms of the 3x3 matrix associated to *Sylvester Algebra*:

$$P = \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix} \quad Q = \Omega \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix} \tag{36}$$

where ω is a cube root of unity and Ω is a cube root of negative unity. In this case φ has three components.

As an example of related mathematical problems, we can consider the general Cauchy problem in the space $LF = L(R^+)\text{x}F(R)$ of functions whose Laplace and Fourier transforms exist.

$$^C D_t^\alpha u(t,x) - \lambda^L D_x^\beta u(t,x) = 0, \quad t > 0, x \in \mathbb{R}, 0 < \alpha \leq 1, \beta > 0 \tag{37}$$

$$\lim_{x \to \pm\infty} u(t,x) = 0, \quad u(0+,x) = g(x) \tag{38}$$

where $^C D_t^\alpha$ is the Caputo fractional partial derivative, which is defined as

$$D_t^\alpha u(t,x) = {}^C D_t^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(\tau,x)}{(t-\tau)^\alpha} d\tau \tag{39}$$

and where D_x^β is the Liouville fractional partial derivative

$$D_x^\beta u(t,x) = {}^L D_x^\beta u(t,x) = \frac{1}{\Gamma(m-\beta)} \frac{\partial^m}{\partial x^m} \int_{-\infty}^x \frac{u(t,z)}{(x-z)^{\beta-m+1}} dz \tag{40}$$

with $m = [\beta]$.

The solution of the Cauchy problem is:

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(k) E_\alpha(\lambda(-ik)^\beta t^\alpha) e^{-ikx} dk \tag{41}$$

where $G(k)$ is the Fourier transform of $g(x)$ and E_α is the Mittag-Leffler function on the complex plane, defined as

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)} \tag{42}$$

For example, for $\beta = 1$ and $g(x) = e^{-\mu|x|}$, $\mu > 0$:

$$u(t,x) = e^{-\mu|x|} E_\alpha(-\mu\lambda t^\alpha) \tag{43}$$

And the moments of the fundamental solution ($g(x) = \delta(x)$, $G(k) = 1$) for the case $\beta = 1$ are obtained as

$$< x^n >= \int_{-\infty}^{\infty} x^n u(t,x) dx = (-\lambda t^\alpha)^n \frac{\Gamma(n+1)}{\Gamma(\alpha n + 1)}, \qquad n = 0, 1, 2, ... \tag{44}$$

This last relation leads to think that a possible application of this kind of fractional equations could be the modeling of the movement and the absorption properties of the dust on the Martian atmosphere. So a part of incident energy on the atmosphere is scattered by this dust and it is observed that the coefficient of molecular scattering τ is a function of the wave-length of the radiation [1], [11], in the form:

$$\tau = \frac{\beta}{\lambda^\alpha} \tag{45}$$

where α and β are characteristic parameters of the Martian dust. In this sense, it is interesting to analyze what fractional differential equations could be related to this Angstrom exponent.

Acknowledgments

We thank to Prof. Francesco Carlino for the opportunity to present a panoramic view of our research in the area of Fractional Calculus.

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*Corresponding author, [1]Departamento de Matemática Aplicada, Facultad de Informática, Universidad Complutense de Madrid, 28040 Madrid, Spain, lvazquez@fdi.ucm.es
[2]Área de Matemáticas, Estadística e Investigación operativa, Centro Universitario de la Defensa, Academia General Militar, 50090 Zaragoza, Spain, velascom@unizar.es,
[3]Departamento de Matemática Aplicada a las TT.II., E.T.S.I. Telecomunicación, Universidad Politécnica de Madrid, 28040 Madrid, Spain, s.jimenez@upm.es