Neural Approximations of the Solutions to a Class of Stochastic Optimal Control Problems

Giorgio Gnecco
IMT School for Advanced Studies – Piazza S. Francesco 19, 55100 Lucca – Italy
giorgio.gnecco@imtlucca.it

Marcello Sanguineti
DIBRIS - University of Genova – Via Opera Pia 13, 16145 Genova – Italy
marcello.sanguineti@unige.it

Abstract

The approximate solution of finite-horizon optimal control problems via neural approximations of the optimal closed-loop control functions is investigated. The analysis enhances the potentialities of recent developments in neural-network approximation in the framework of sequential decision problems with continuous state and control spaces. A class of stochastic optimal control problems with bilinear dynamical systems is investigated, for which neural-network approximation mitigates the curse of dimensionality. More specifically, the minimal number of network parameters needed to achieve a desired accuracy of the approximate solution does not grow exponentially with the number of state variables. The results obtained provide a theoretical basis to the development of neural-network-based approaches for the suboptimal control of stochastic dynamical systems.

Keywords: stochastic optimal control, finite horizon, optimal control functions, neural networks, approximation.

1 Introduction

It is well known that often applied sciences and engineering involve problems with a degree of uncertainty. For instance, uncertainty can be associated with stock prices in financial applications, outcomes of market analysis in production planning, rain inflows in water reservoirs systems, lengths of message queues in telecommunication networks, traveling times in traffic management, etc. This is the typical case when one has to deal with problems of scheduling, transportation, production planning and location, inventory, investment, facilities and equipment planning, industrial marketing, management of water resources, asset pricing, capital rationing, taxation, vehicle routing, engineering design, which are naturally formulated in stochastic environments.

Here, we focus on N-stage optimization problems, in which the decisions have to be taken in such a way to maximize the expected value, with respect to the uncertainties, of a reward (or minimize an expected cost) expressed as a summation over a finite number of stages, and the decisions taken at each stage depend on *state variables* that capture the "history" of the optimization process. Unfortunately, in general such problems cannot be solved in closed form. Hence, one has to search for suboptimal solutions. Experimental results have shown that optimization over decision functions built from relatively few computational units with a simple structure may obtain surprisingly good performance in high-dimensional optimization tasks (see, e.g., [8, 14, 16, 17, 20, 22] and the references therein). In these models, the closed-loop decision functions are made up as linear combinations of input-output maps computed by units belonging to some *dictionary* [9, 10, 13]. Examples of dictionaries are those made by perceptron, radial or kernel units, Hermite functions, trigonometric polynomials, and splines. When the elements of the dictionary are nonlinearly parameterized, one has nonlinear approximation schemes, such as those based on neural networks [15].

We investigate from a theoretical point of view some issues about the application of nonlinear approximation schemes of the neural-network type to find accurate suboptimal closed-loop control functions in a class of stochastic optimal control problems. We derive upper bounds on the loss in performance when the optimal closed-loop control functions are replaced by their approximations, and we investigate cases for which upper bounds on such a loss can be translated into upper bounds on the approximation error of the optimal closed-loop control function at each stage. The departure points of our analysis are [6, 7, 12]. Results in the flavour of our Theorems 3.1 and 3.2 were derived therein, but for a different optimization model, in which the transition between pairs of states is described by a correspondence, rather than a state equation. In particular, Theorem 3.1 differs, e.g., from [7, Lemma 17], since the former is stated for bilinear stochastic dynamical systems. This class of systems is important since they are the most direct generalization of linear systems. Moreover, Theorem 3.3 investigates how upper bounds on the errors in approximate optimization translate into upper bounds on the approximation errors of the optimal closed-loop control functions.

The paper is organized as follows. Section 2 describes the model of finite-horizon stochastic optimal control problems that we address. Section 3 relates bounds on the approximation error of the optimal closed-loop control functions to bounds on the loss in performance associated with the use of their approximations. It also presents a smoothness result, needed for the application of such bounds. Section 4 specializes the analysis to neural-network-based approximations of the optimal closed-loop control functions, investigating cases for which they mitigate the curse of dimensionality (i.e., such that the minimal number of network parameters required to achieve a desired accuracy of the suboptimal solutions does not grow exponentially with the number of state variables). Finally, Section 5 concludes the paper with a discussion. The Appendix contains some technical proofs.

2 The model of stochastic optimal control

We consider a dynamical system described by the state equation

$$x_{t+1} = f_t(x_t, u_t, \xi_t)\,, \quad t = 0, 1, \ldots, N-1, \tag{1}$$

where $x_t \in X_t \subseteq \mathbb{R}^d$ is a continuous state vector, $x_0 = \hat{x} \in X_0$ is a given initial state, $u_t \in U_t \subseteq \mathbb{R}^m$ is a continuous control vector, and $\xi_t \in \Xi_t \subseteq \mathbb{R}^r$ are mutually independent random vectors. The sets X_t satisfy the constraints $X_{t+1} = \{y \in \mathbb{R}^d : y = f_t(x_t, u_t, \xi_t), x_t \in X_t, u_t \in U_t, \xi_t \in \Xi_t\}$, for $t = 0, \ldots, N-1$. We denote by $g_t : X_t \mapsto U_t$ the admissible closed-loop control functions, where x_t is assumed to be known to the decision maker at time stage t.

We state the following stochastic optimal control problem.

Problem SOCP. *Find a sequence of optimal closed-loop control functions $g_0^\circ, \ldots, g_{N-1}^\circ$ that minimizes the cost functional*

$$J := \mathbb{E}_{\xi_0,\ldots,\xi_{N-1}} \left\{ \sum_{t=0}^{N-1} h_t(x_t, g_t(x_t), \xi_t) + h_N(x_N) \right\} \tag{2}$$

subject to the constraints $x_t \in X_t, u_t \in U_t$, and (1).

Let us define the optimal cost-to-go functions

$$J_t^\circ(x_t) := \min_{g_t,\ldots,g_{N-1}} \mathbb{E}_{\xi_t,\ldots,\xi_{N-1}} \left\{ \sum_{k=t}^{N-1} h_k(x_k, g_k(x_k), \xi_k) + h_N(x_N) \right\},$$

for $t = 0, \ldots, N-1$. Then, the Dynamic Programming (DP) recursive equation (which holds under mild conditions, see Assumption 3.1 (i-iii) later) is given by

$$\begin{aligned} J_N^\circ(x_N) &= h_N(x_N)\,, \\ J_t^\circ(x_t) &= \min_{u_t \in U_t} \mathbb{E}_{\xi_t} \left\{ h_t(x_t, u_t, \xi_t) + J_{t+1}^\circ(x_{t+1}) \right\} \\ & \quad t = N-1, \ldots, 0\,. \end{aligned} \tag{3}$$

We denote by $\tilde{g}_0, \ldots, \tilde{g}_{N-1}$ some approximations of the optimal closed-loop control functions $g_0^\circ, \ldots, g_{N-1}^\circ$. Using such approximate control functions, we now consider approximations of the optimal cost-to-go functions of the form

$$\tilde{J}_t(x_t) := \mathbb{E}_{\xi_t,\ldots,\xi_{N-1}} \left\{ \sum_{k=t}^{N-1} h_k(x_k, \tilde{g}_k(x_k), \xi_k) + h_N(x_N) \right\}, \tag{4}$$

for $t = 0, \ldots, N-1$. It is worth noting that the approximations (4) of the optimal cost-to-go functions J_t° are obtained indirectly through the vector-valued approximations $\tilde{g}_t, \ldots, \tilde{g}_{N-1}$, while in Approximate Dynamic Programming [21] a single scalar-valued function is used to approximate J_t° directly.

Finally, for a bounded and continuous function $s_t : X_t \mapsto \mathbb{R}^m$, $s_t = \text{col}\,(\, s_{t,1}, \ldots, s_{t,m}\,)$, we let

$$\|s_t\|_{\sup(X_t)} := \sup_{x_t \in X_t} \|s_t(x_t)\| := \sup_{x_t \in X_t} \sqrt{\sum_{j=1}^{m} s_{t,j}^2(x_t)}\,. \tag{5}$$

3 Accuracies of suboptimal solutions

In this section we present theoretical results that investigate the approximate solution of Problem SOCP. Then, in Section 4, we specialize such results to the case of neural-network-based approximation of the optimal closed-loop control functions.

The next Theorem 3.1 (reported, without proof, in [5]) guarantees, for each fixed stage and state vector, the continuity of the associated cost-to-go functional with respect to some closed-loop control functions, using the supremum norm. For the moment, let us use for the the approximations $\tilde{J}_t(x_t)$ the extended notation $\tilde{J}_t(x_t, \tilde{g}_t, \ldots, \tilde{g}_{N-1})$, to emphasize their dependence on $\tilde{g}_t, \ldots, \tilde{g}_{N-1}$. Then, Theorem 3.1 shows that, for each $x_t \in X_t$, the cost-to-go functional $\tilde{J}_t(x_t, \tilde{g}_t, \ldots, \tilde{g}_{N-1})$ is continuous with respect to $\tilde{g}_t, \ldots, \tilde{g}_{N-1}$ when the supremum norm is used to measure the approximation errors of the optimal closed-loop control functions. Of course, when considering the optimal closed-loop control functions, one obtains $\tilde{J}_t(x_t, g_t^\circ, \ldots, g_{N-1}^\circ) = J_t^\circ(x_t)$. The choice of the supremum norm in Theorem 3.1 is motivated by the fact that a similar result cannot be obtained, e.g., if the $\mathcal{L}^2$-norm is used for the same purpose (unless the sets X_t are compact, and suitable smoothness constraints are made on the family of approximators).

Theorem 3.1 *For $t = 0, \ldots, N-1$, suppose that there exist optimal closed-loop control functions g_t°, and let f_t, h_t, h_N, and g_t° be bounded and Lipschitz continuous, with Lipschitz constants bounded from above by L_{f_t}, L_{h_t}, L_{h_N} and $L_{g_t^\circ}$, respectively. Let $\tilde{g}_t : X_t \mapsto U_t$ be bounded and continuous approximating closed-loop control functions such that $\|g_t^\circ - \tilde{g}_t\|_{\sup(X_t)} \le \epsilon_t$ for some $\epsilon_t \ge 0$. Then, for each $x_t \in X_t$, $\tilde{J}_t$ is continuous with respect to $\tilde{g}_t, \ldots, \tilde{g}_{N-1}$ in the supremum norm and*

$$\|J_t^\circ - \tilde{J}_t\|_{\sup(X_t)} \le \sum_{k=t}^{N-1} \Psi_k\, \epsilon_k\,, \tag{6}$$

where, for $k = t, \ldots, N-1$, $\Psi_k = \big(\sum_{j=k}^{N-1} L_{h_j}\big((1 + L_{g_k^\circ})\Theta_{jk} + \delta_{jk}\,\big)\big) + L_{h_N}\Theta_{Nk}$, $\Theta_{kk} = 0$, $\Theta_{k+1,k} = L_{f_k}$, $\Theta_{k+i,k} = \big(\prod_{j=k+1}^{k+i-1}(1 + L_{g_j^\circ})L_{f_j}\big)L_{f_k}$, $i = 2, \ldots, N-k$, and δ_{jk} is the Kronecker's delta.

The next Theorem 3.2 provides a smoothness result valid for the optimal closed-loop control functions and the optimal cost-to-go functions. The result is an adaptation to the case of bilinear stochastic dynamical systems of [7, Lemma 17], which was stated in terms of correspondences. It can be exploited, e.g., to guarantee the Lipschitz continuity assumption of the optimal closed-loop control functions, made in Theorem 3.1. Moreover, it is needed to prove the next Theorem 3.3, where conditions are presented under which one can translate an upper bound on the loss in performance due to the approximation of the optimal closed-loop control functions into an upper bound on the approximation error of the optimal closed-loop control function at a given stage. It will be also exploited to obtain the results derived in Section 4, where the performance of neural-network-based approximators is investigated, when they are used to construct suboptimal solutions to Problem SOCP.

We use the following notations for the partial derivatives. By ∇ we denote the nabla operator. When applied to a scalar function, it gives its gradient (defined here as a column vector). When applied to a vector function, it returns its Jacobian, i.e., the matrix whose rows are the transposes of the gradients of the components of the function. Let d_1, d_2, and d_3 be positive integers, $x \in \mathbb{R}^{d_1}$, $y \in \mathbb{R}^{d_2}$, and $z \in \mathbb{R}^{d_3}$. In the case of a composite function, e.g., $f(g(x,y,z), h(x,y,z))$, $\nabla_1 f(g(x,y,z), h(x,y,z))$ denotes the vector of partial derivatives of f with respect to its first vector argument, computed at $(g(x,y,z), h(x,y,z))$, and $\nabla_x f(g(x,y,z), h(x,y,z))$ the full gradient of f with respect to x. ∇^2 denotes the *Hessian*:

$$\nabla^2 f(x,y,z) = \begin{pmatrix} \nabla^2_{1,1} f(x,y,z) & \nabla^2_{1,2} f(x,y,z) & \nabla^2_{1,3} f(x,y,z) \\ \nabla^2_{2,1} f(x,y,z) & \nabla^2_{2,2} f(x,y,z) & \nabla^2_{2,3} f(x,y,z) \\ \nabla^2_{3,1} f(x,y,z) & \nabla^2_{3,2} f(x,y,z) & \nabla^2_{3,3} f(x,y,z) \end{pmatrix}, \tag{7}$$

where for every $(x,y,z) \in \mathbb{R}^{(d_1+d_2+d_3)}$, $\nabla^2 f(x,y,z) \in \mathbb{R}^{(d_1+d_2+d_3)\times(d_1+d_2+d_3)}$. For a non-negative integer s, by $\mathcal{C}^s$ we denote the class of scalar-valued or vector-valued functions (depending on the context) that are continuously differentiable up to the order s. For a set $X \subseteq \mathbb{R}^l$, we denote by $\mathrm{int}(X)$ its interior. Finally, for a symmetric real matrix M, we denote by $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ its minimum and maximum eigenvalues, respectively.

In the next assumption, items (i-iii) express basic compactness, continuity and convexity requirements under which we study Problem SOCP, whereas items (iv) and (v) impose some form of strong convexity on the transition and final cost functions. Item (vi) is an assumption of continuity and interiority of the optimal closed-loop control functions, whereas item (vii) is an assumption on the transition function f_t.

Assumption 3.1 *Let $s \geq 2$ be an integer, and the following hold.*
(i) For $t = 0, \ldots, N-1$, $\Xi_t \subset \mathbb{R}^r$ is compact and has nonempty interior, and each ξ_t has a probability density $p_t(\xi_t) \in \mathcal{C}^s(\Xi_t)$.
(ii) For $t = 0, \ldots, N$, $X_t \subset \mathbb{R}^d$ is compact and convex, and has nonempty interior. For $t = 0, \ldots, N-1$, $U_t \subset \mathbb{R}^m$ is compact and convex, and has nonempty interior.
(iii) For $t = 0, \ldots, N-1$, $f_t \in \mathcal{C}^s(X_t \times U_t \times \Xi_t)$.

(iv) $h_N \in \mathcal{C}^s(X_N)$*, and there exists* $\alpha_N > 0$ *such that one has*

$$\inf_{x_N \in X_N} \lambda_{\min}(\nabla^2 h_N(x_N) \geq \alpha_N \,.$$

(v) For $t = 0, \dots, N-1$*,* $h_t \in \mathcal{C}^s(X_t \times U_t \times \Xi_t)$*, and for every* $\xi_t \in \Xi_t$ *the function* $h_t(\cdot,\cdot,\xi_t)$ *is convex, and there exists* $\alpha_t > 0$ *such that one has*

$$\inf_{x_t \in X_t, u_t \in U_t, \xi_t \in \Xi_t} \lambda_{\min}(\nabla^2_{2,2} h_t(x_t, u_t, \xi_t)) \geq \alpha_t \,.$$

(vi) There exist continuous optimal closed-loop control functions $g_0^\circ(x_0), \dots, g_{N-1}^\circ(x_{N-1})$ *and they are interior for every* $x_t \in \mathrm{int}(X_t)$*, i.e.,* $g_t^\circ(x_t) \in \mathrm{int}(U_t)$ *for* $x_t \in \mathrm{int}(X_t)$*.*
(vii) For $t = 0, \dots, N-1$*, the function* $f_t \in \mathcal{C}^s(X_t, U_t, \Xi_t)$ *is affine in the control* u_t*, i.e., it has the form* $f_t(x_t, u_t, \xi_t) = A_t(x_t, \xi_t) u_t + b_t(x_t, \xi_t)$ *(where* $A_t(x_t, \xi_t)$ *and* $b_t(x_t, \xi_t)$ *are real* $d \times m$ *and* $d \times 1$ *matrices, respectively). Moreover, for* $t = 0, \dots, N-1$*, there exists* $\eta_t \geq 0$ *such that*

$$\inf_{x_t \in X_t, \xi_t \in \Xi_t} \lambda_{\min}\left(A_t^T(x_t, \xi_t) A_t(x_t, \xi_t)\right) \geq \eta_t \,.$$

Finally, the matrices $A_t(x_t, \xi_t)$ *and* $b_t(x_t, \xi_t)$ *are affine in the state* x_t*.*

Assumption 3.1 (vii) implies that the stochastic dynamical system to be controlled is bilinear in x_t and u_t. Assumption 3.1 is satisfied, e.g., by suitable bilinear "perturbations" of the classical Linear Quadratic (LQ) optimal control problem, constructed likewise in [12, Section 6]. Moreover, it also holds for suitable dynamical and bilinear "perturbations" of static optimization problems, obtained by inserting a sufficiently small discount factor β in the model. In both cases, indeed, the optimal closed-loop control functions of the perturbed problem are similar to the ones of the simpler original problem, which can be often solved in closed form. So, if one imposes Assumption 3.1 on the original problem, one can then define the perturbed problem in such a way that it also satisfies the same assumption.

Theorem 3.2 *Let Assumption 3.1 hold. Then, for* $t = 0, \dots, N$*,* $J_t^\circ \in \mathcal{C}^s(X_t)$ *and*

$$\inf_{x_t \in X_t} \lambda_{\min}(\nabla^2 (J_t^\circ(x_t))) \geq \alpha_t \,. \tag{8}$$

Moreover, for $t = 0, \dots, N-1$*,* $g_t^\circ \in \mathcal{C}^{s-1}(X_t)$*.*

The next Theorem 3.3 states that, if suitable strong convexity conditions hold (together with other conditions), then finding a good approximation of the minimum of the functional $\tilde{J}_t(x_t, \tilde{g}_t, \dots, \tilde{g}_{N-1})$ allows one to obtain a good approximation of its minimizer. More precisely, it shows that, under suitable assumptions, if one approximately minimizes $\tilde{J}_{N-1}(x_{N-1}, \tilde{g}_{N-1})$ for each $x_{N-1} \in X_{N-1}$, then the resulting approximate minimizer $\tilde{g}_{N-1}^\circ$ is a good approximation of the optimal closed-loop control function g_{N-1}°. A similar result holds for the other stages $t = N-2, \dots, 0$.

Theorem 3.3 *Let Assumption 3.1 hold. Then, for every $\varepsilon_{N-1} > 0$ one has the following implication:*

$$\begin{aligned} & \|\tilde{J}_{N-1}(\cdot, g^\circ_{N-1}) - \tilde{J}_{N-1}(\cdot, \tilde{g}^\circ_{N-1})\|_{\sup(X_{N-1})} \leq \varepsilon_{N-1} \\ \Rightarrow \quad & \|g^\circ_{N-1} - \tilde{g}^\circ_{N-1}\|_{\sup(X_{N-1})} \leq \sqrt{\tfrac{2\varepsilon_{N-1}}{\alpha_{N-1}+\alpha_N \eta_{N-1}}}\,. \end{aligned} \tag{9}$$

Moreover, for any $t = N-2, \ldots, 0$, and every $\varepsilon_t, \varepsilon_{t+1} > 0$ one has the following implication:

$$\begin{aligned} & \|\tilde{J}_t(\cdot, g^\circ_t, \ldots, g^\circ_{N-1}) - \tilde{J}_t(\cdot, \tilde{g}^\circ_t, \ldots, \tilde{g}^\circ_{N-1})\|_{\sup(X_t)} \leq \varepsilon_t \\ \text{and} \quad & \|\tilde{J}_{t+1}(\cdot, g^\circ_{t+1}, \ldots, g^\circ_{N-1}) - \tilde{J}_{t+1}(\cdot, \tilde{g}^\circ_{t+1}, \ldots, \tilde{g}^\circ_{N-1})\|_{\sup(X_{t+1})} \leq \varepsilon_{t+1} \\ \Rightarrow \quad & \|g^\circ_t - \tilde{g}^\circ_t\|_{\sup(X_t)} \leq \sqrt{\tfrac{2(\varepsilon_t+\varepsilon_{t+1})}{\alpha_t+\alpha_{t+1}\eta_t}}\,. \end{aligned} \tag{10}$$

4 Suboptimal solutions via neural networks

In this section, we investigate the application of neural-network-based approximation schemes with sigmoidal computational units to the approximate solution of Problem SOCP.

Definition 4.1 *A* sigmoidal function $\sigma : \mathbb{R} \to \mathbb{R}$ *is a bounded measurable function such that $\sigma(y) \to 1$ as $y \to +\infty$, and $\sigma(y) \to 0$ as $y \to -\infty$ [2]).*

We first report an upper bound on the function approximation error by sigmoidal neural networks, for a specific choice of the set of functions to be approximated. The following Theorem 4.1 from [1] describes a quite general set of functions of d real variables (described in terms of their Fourier distributions) whose approximation from neural-network-based approximation schemes with sigmoidal computational units requires[1] $O\left(\varepsilon^{-2}\right)$ computational units, where $\varepsilon > 0$ is the desired worst-case approximation error measured in the supremum norm. In the following, we denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{R}^d$.

Theorem 4.1 *Let $C > 0$, d a positive integer, B a bounded subset of $\mathbb{R}^d$ containing 0, and $\Gamma_{B,C}$ the set of functions $f : \mathbb{R}^d \to \mathbb{R}$ having a Fourier representation of the form*

$$f(x) = \int_{\mathbb{R}^d} e^{i\langle \omega, x\rangle} \hat{F}(d\omega) \tag{11}$$

for some complex-valued measure $\hat{F}(d\omega) = e^{i\theta(\omega)} F(d\omega)$ (where $F(d\omega)$ and $\theta(\omega)$ are the magnitude distribution and the phase at the pulsation ω, respectively) such that

$$\int_{\mathbb{R}^d} \sup_{x \in B} |\langle \omega, x\rangle| F(d\omega) \leq C\,. \tag{12}$$

[1]For two functions $f, g : (0, +\infty) \to \mathbb{R}$, one writes $f = O(g)$ if and only if there exist $M > 0$ and $x_0 > 0$ such that $f(x) \leq M g(x)$ for all $x \in (0, x_0)$. In order to use this notation also for multivariable functions, it is assumed that all their arguments are fixed, with the exception of one of them (here, ε).

Then, for every $n \geq 1$, there exist $a_1, \ldots, a_n \in \mathbb{R}^d$, $b_1, \ldots, b_n, c_0, c_1, \ldots, c_n \in \mathbb{R}$, and $f_n : B \to \mathbb{R}$ of the form

$$f_n(x) = \sum_{k=1}^{n} c_k \sigma(\langle a_k, x\rangle + b_k) + c_0 \tag{13}$$

(where $\sigma : \mathbb{R} \to \mathbb{R}$ is a sigmoidal function) such that

$$\sup_{x \in B} |f(x) - f_n(x)| \leq \frac{120C}{\sqrt{n}} d\,. \tag{14}$$

Similar upper bounds on the approximation error expressed in the supremum norm, and valid for other neural-network-based approximation schemes (including those with either Gaussian or compactly supported computational units), are given, e.g., in [11, 19, 24]. The next Theorem 4.2 shows that under Assumption 3.1, if the degree of smoothness s is sufficiently large compared with the dimension of the state space, then the optimal closed-loop control functions can be approximated by a neural-network-based approximation scheme of the form (13), without incurring the curse of dimensionality in the minimal number of parameters needed by the approximation scheme to achieve a desired accuracy. For simplicity, here only the case of sigmoidal computational units is presented, but the result can be extended to the approximation schemes considered, e.g., in [11,19,24].

Theorem 4.2 *Let Assumption 3.1 hold with $s \geq \lfloor \frac{d}{2}+3 \rfloor$. Then, for each stage t and each component $g^{\circ}_{t,j}$ of the t-th optimal closed-loop control function, there exists a constant $C_{t,j} > 0$ such that, for every positive integer n, there is a function $f_{t,j,n}$ of the form (13) for which*

$$\sup_{x_t \in X_t} |g^{\circ}_{t,j}(x_t) - f_{t,j,n}(x_t)| \leq \frac{120C_{t,j}}{\sqrt{n}} d\,. \tag{15}$$

Proof. It follows from Theorem 3.2 that $g^{\circ}_{t,j} \in \mathcal{C}^{s-1}(X_t)$, where X_t is bounded and convex. Then, by Sobolev's extension theorem [23, Theorem 5, p. 181, and Example 2, p. 189], it can be extended to a function $\bar{g}^{\circ}_{t,j}$ belonging to the Sobolev space[2] $\mathcal{W}^{s-1,2}(\mathbb{R}^d)$, hence it also belongs to the set $\Gamma_{X_t, C_{t,j}}$ for a sufficiently large $C_{t,j} > 0$, since $s - 1 \geq \lfloor \frac{d}{2} + 2 \rfloor$ (see [2, Section IX, Example 15]). Finally, one concludes by applying Theorem 4.1. □

The following result provides an upper bound on the loss of performance in solving Problem SOCP, when functions of the form (13) are used to approximate the components of the optimal closed-loop control functions.

Theorem 4.3 *Let Assumption 3.1 hold with $s \geq \lfloor \frac{d}{2} + 3 \rfloor$. Then, there exist constants $C_{t,j} > 0$ such that the following hold. Let each component $\tilde{g}_{t,j}$ of the approximation $\tilde{g}_t$ of each optimal closed-loop control function g°_t be of the form (13), with $n = n_{t,j}$. Then, there exists one choice for the functions*

[2] I.e., the space of locally integrable functions $f : \mathbb{R}^d \to \mathbb{R}$ whose weak partial derivatives up to the order $s - 1$ are square-integrable.

$\tilde{g}_{t,j}$ *for which*

$$\|J_t^\circ - \tilde{J}_t\|_{\sup(X_t)} \leq \sum_{k=t}^{N-1} \Psi_k \, \Phi \sqrt{\sum_{j=1}^{m} \frac{C_{k,j}^2}{n_{k,j}}} \,, \tag{16}$$

where the constants Ψ_k *are defined as in Theorem 3.1, and* $\Phi = 120\, d$.

Proof. The result is obtained by combining Theorems 3.1 and 4.3 (the Lipschitz continuity assumptions of Theorem 3.1 follow from Assumption 3.1 (iv,v,vii) and from Theorem 3.2), and by using (5). □

5 Discussion

Theorems 3.1 and 3.3 can be exploited to investigate the relationship between the error in the approximation of the optimal closed-loop control functions in stochastic optimal control problems, and the loss in performance due to such an approximation. Theorem 3.1 does this in a "forward" manner (showing how upper bounds on the approximation errors of the optimal closed-loop control functions can be translated into upper bounds on the loss in performance), while Theorem 3.3 works in a "backward" manner (showing how upper bounds on the loss in performance can be translated into upper bounds on the approximation error).

Theorem 4.2 shows that, under suitable assumptions, the optimal closed-loop control functions can be approximated by using a sigmoidal approximation scheme, without incurring the curse of dimensionality in the minimal number of parameters used by such an approximation scheme. Theorem 4.3 shows how the loss in performance can be controlled, again without incurring the curse of dimensionality.

The analysis can be extended to other neural-network-based approximation schemes, using the upper bounds on the function approximation error provided in [11, 19, 24]. As another possible extension, our results (particularly, Theorem 3.3) can be applied to linear approximation schemes, to find cases for which the approximate solution of Problems SOCP through such schemes suffers, instead, from the curse of dimensionality. In particular, this extension could be obtained by combining Theorem 3.3 with the inverse problem technique that was already used in [3, 4] to construct dynamic optimization problems associated with given optimal policy functions, choosing such functions in such a way that they are "easily" approximable by sigmoidal neural networks, but "hard" to be approximated by linear approximation schemes.

Appendix

Here, we collect the proofs of some technical results.

Proof of Theorem 3.1. For $x_t \in X_t$, let $x_t^\circ = \tilde{x}_t = x_t$, and, for $k = t, \ldots, N-1$, under the same realizations of the random vectors ξ_k, let

$$x_{k+1}^\circ := f_k(x_k^\circ, g_k^\circ(x_k^\circ), \xi_k)\,, \tag{17}$$

and

$$\tilde{x}_{k+1} := f_k(\tilde{x}_k, \tilde{g}_k(\tilde{x}_k), \xi_k)\,. \tag{18}$$

By the Lipschitz continuity of the functions h_k and h_N and the triangle inequality, we get

$$\|J_t^\circ - \tilde{J}_t\|_{\sup(X_t)} \leq \mathbb{E}_{\xi_t,\ldots,\xi_N}\left\{ \sum_{k=t}^{N-1} L_{h_k}(\|x_k^\circ - \tilde{x}_k\| + \|g_k^\circ(x_k^\circ) - \tilde{g}_k(\tilde{x}_k)\|) + L_{h_N}\|x_N^\circ - \tilde{x}_N\| \right\}. \tag{19}$$

The Lipschitz continuity of f_k and g_k°, the triangle inequality, and the definition of $\tilde{g}_k$ give

$$\begin{aligned}
& \|g_k^\circ(x_k^\circ) - \tilde{g}_k(\tilde{x}_k)\| \\
= \;& \|g_k^\circ(x_k^\circ) - g_k^\circ(\tilde{x}_k) + g_k^\circ(\tilde{x}_k) - \tilde{g}_k(\tilde{x}_k)\| \\
\leq \;& \|g_k^\circ(x_k^\circ) - g_k^\circ(\tilde{x}_k)\| + \|g_k^\circ(\tilde{x}_k) - \tilde{g}_k(\tilde{x}_k)\| \\
\leq \;& L_{g_k^\circ}\|x_k^\circ - \tilde{x}_k\| + \|g_k^\circ(\tilde{x}_k) - \tilde{g}_k(\tilde{x}_k)\| \\
\leq \;& L_{g_k^\circ}\|x_k^\circ - \tilde{x}_k\| + \epsilon_k\,,
\end{aligned} \tag{20}$$

and by (20)

$$\begin{aligned}
& \|x_{k+1}^\circ - \tilde{x}_{k+1}\| \\
\leq \;& L_{f_k}\sqrt{\|x_k^\circ - \tilde{x}_k\|^2 + \|g_k^\circ(x_k^\circ) - \tilde{g}_k(\tilde{x}_k)\|^2} \\
\leq \;& L_{f_k}\left(\|x_k^\circ - \tilde{x}_k\| + \|g_k^\circ(x_k^\circ) - \tilde{g}_k(\tilde{x}_k)\|\right) \\
\leq \;& L_{f_k}(1 + L_{g_k^\circ})\|x_k^\circ - \tilde{x}_k\| + L_{f_k}\epsilon_k\,.
\end{aligned} \tag{21}$$

Let $\theta_t = 0$ and define the sequence $\theta_{k+1} = L_{f_k}(1 + L_{g_k^\circ})\theta_k + L_{f_k}\epsilon_k$. Let $\tilde{\theta}_t = 0$ and $\tilde{\theta}_k = \|x_k^\circ - \tilde{x}_k\|$. Then, the last upper bound in formula (21) provides $\tilde{\theta}_{k+1} \leq L_{f_k}(1 + L_{g_k^\circ})\tilde{\theta}_k + L_{f_k}\epsilon_k$, and also $\theta_k \geq \tilde{\theta}_k$. The proof is completed by applying (19), solving the linear difference equation $\theta_{k+1} = L_{f_k}(1 + L_{g_k^\circ})\theta_k + L_{f_k}\epsilon_k$, and regrouping the terms multiplying the same ϵ_k. □

Before presenting the proof of Theorem 3.2, we report the following technical lemma (which follows directly from [25, Theorem 2.13, p. 69] and from the example in [25, p. 70]), since it is used in its proof. We recall that, given a square partitioned real matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{22}$$

such that D is non-singular, *Schur's complement* M/D of D in M is defined [25, p. 18] as the matrix

$$M/D := A - BD^{-1}C\,. \tag{23}$$

Lemma 5.1 *Let* $M = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$ *be a partitioned symmetric positive-semidefinite matrix such that* D *is non-singular. Then* $\lambda_{\min}(M/D) \geq \lambda_{\min}(M)$.

Proof of Theorem 3.2. Assumption 3.1 (i-iii) ensures that, by an application of [18, Theorem 2.6], every t-th optimal cost-to-go function J_t° satisfies the DP recursive equation (3).

For $t = N$, $J_t^\circ = h_N \in \mathcal{C}^s(X_t)$ and satisfies

$$\inf_{x_t \in X_t} \lambda_{\min}(\nabla^2(J_t^\circ(x_t))) \geq \alpha_t\,,$$

due to the first formula (3) and Assumption 3.1 (iv). For $t = N-1, \ldots, 0$, the fact that $J_t^\circ \in \mathcal{C}^s(X_t)$ and satisfies (8) is proved by the following backward induction argument. As a by-product of such an argument, we also show that $g_t^\circ \in \mathcal{C}^{s-1}(X_t)$.

Let $x_t \in \mathrm{int}(X_t)$. Since by Assumption 3.1 (vi) the optimal closed-loop control function g_t° is continuous and interior on $\mathrm{int}(X_t)$, the first-order optimality condition

$$\mathbb{E}_{\xi_t}\left\{\nabla_2 h_t(x_t, g_t^\circ(x_t), \xi_t) + (\nabla_2 f_t(x_t, g_t^\circ(x_t)), \xi_t)^T\, \nabla J_{t+1}^\circ(f_t(x_t, g_t^\circ(x_t), \xi_t))\right\} = 0 \tag{24}$$

holds. By differentiating (24) and omitting the arguments (in order to simplify the notation), we obtain

$$\begin{aligned}\mathbb{E}_{\xi_t}\Big\{&\nabla^2_{2,1} h_t + [(\nabla_2 f_t)^T\, \nabla^2 J^\circ_{t+1}\, \nabla_1 f_t + \nabla J^\circ_{t+1} \cdot \nabla^2_{2,1} f_t] \\ &+ \left[\nabla^2_{2,2} h_t + [(\nabla_2 f_t)^T\, \nabla^2 J^\circ_{t+1}\, \nabla_2 f_t + \nabla J^\circ_{t+1} \cdot \nabla^2_{2,2} f_t]\,\right] \nabla g_t^\circ \Big\} = 0\,,\end{aligned} \tag{25}$$

where $\nabla J^\circ_{t+1} \cdot \nabla^2_{2,1} f_t$ and $\nabla J^\circ_{t+1} \cdot \nabla^2_{2,2} f_t$ denote the inner products between the corresponding vectors and third-order tensors, i.e., using the index j to denote the components of x_t and f_t,

$$\nabla J^\circ_{t+1} \cdot \nabla^2_{2,1} f_t = \sum_{j=1}^{d} \frac{\partial J^\circ_{t+1}}{\partial x_{t+1,j}} \nabla^2_{2,1} f_{t,j} \tag{26}$$

and

$$\nabla J^\circ_{t+1} \cdot \nabla^2_{2,2} f_t = \sum_{h=1}^{d} \frac{\partial J^\circ_{t+1}}{\partial x_{t+1,j}} \nabla^2_{2,2} f_{t,j}\,. \tag{27}$$

We set (omitting the arguments at the right-hand side)

$$N_t(x_t) := \mathbb{E}_{\xi_t}\left\{\nabla^2_{2,1} h_t + [(\nabla_2 f_t)^T\, \nabla^2 J^\circ_{t+1}\, \nabla_1 f_t + \nabla J^\circ_{t+1} \cdot \nabla^2_{2,1} f_t]\right\} \tag{28}$$

and

$$D_t(x_t) := \mathbb{E}_{\xi_t}\left\{\nabla^2_{2,2}h_t + [(\nabla_2 f_t)^T\,\nabla^2 J^\circ_{t+1}\,\nabla_2 f_t + \nabla J^\circ_{t+1}\cdot\nabla^2_{2,2}f_t]\right\}\,, \tag{29}$$

Note that, since by Assumption 3.1 (vii) f_t is affine in the control u_t, one has $\nabla^2_{2,2}f_t = 0$, so the matrix $D_t(x_t)$ has the simplified expression

$$D_t(x_t) = \mathbb{E}_{\xi_t}\left\{\nabla^2_{2,2}h_t + [(\nabla_2 f_t)^T\,\nabla^2 J^\circ_{t+1}\,\nabla_2 f_t]\right\} \tag{30}$$

and is non-singular, since its minimum eigenvalue $\lambda_{\min}(D_t(x_t))$ satisfies

$$\lambda_{\min}(D_t(x_t)) \geq \alpha_t + \alpha_{t+1}\eta_t \geq \alpha_t > 0 \tag{31}$$

by Assumption 3.1 (v,vii) and

$$\inf_{x_{t+1}\in X_{t+1}} \lambda_{\min}(\nabla^2(J^\circ_{t+1}(x_{t+1}))) \geq \alpha_{t+1}$$

(which holds by the backward induction hypothesis). So, one can apply the implicit function theorem to g°_t, thus obtaining

$$\nabla g^\circ_t(x_t) = -D_t^{-1}(x_t)N_t(x_t) \tag{32}$$

and $g^\circ_t \in \mathcal{C}^{s-1}(\mathrm{int}(X_t))$, still by the implicit function theorem. As the expressions that one can obtain from (32) for its partial derivatives up to the order $s-1$ are bounded and continuous not only on $\mathrm{int}(X_t)$ but on the whole X_t, one has $g^\circ_t \in \mathcal{C}^{s-1}(X_t)$.

Similarly, using the DP recursive equation (3) and omitting the arguments at the right-hand side, one obtains the following expression for the gradient of J°_t:

$$\nabla J^\circ_t(x_t) = \mathbb{E}_{\xi_t}\left\{\nabla_1 h_t + (\nabla_1 f_t)^T\,\nabla J^\circ_{t+1}\right\}\,, \tag{33}$$

where we have exploited condition (24) to cancel out the zero contribution of $\nabla g^\circ_t(x_t)$ to (33). By differentiating the two members of (33) up to derivatives of h_t of order s, we obtain $J^\circ_t \in \mathcal{C}^s(\mathrm{int}(X_t))$. Likewise for the optimal closed-loop control functions, this extends to $J^\circ_t \in \mathcal{C}^s(X_t)$.

Finally, by setting

$$M_t(x_t) := \begin{pmatrix} \mathbb{E}_{\xi_t}\left\{\nabla^2_{1,1}h_t + (\nabla_1 f_t)^T\nabla^2 J^\circ_{t+1}\nabla_1 f_t\right\} & \mathbb{E}_{\xi_t}\left\{\nabla^2_{1,2}h_t + (\nabla_1 f_t)^T\nabla^2 J^\circ_{t+1}\nabla_2 f_t\right\} \\ \mathbb{E}_{\xi_t}\left\{\nabla^2_{2,1}h_t + (\nabla_2 f_t)^T\nabla^2 J^\circ_{t+1}\nabla_1 f_t\right\} & D_t(x_t) \end{pmatrix} \tag{34}$$

and using (32) and (33) and the definition of the Schur's complement, one obtains the following expression for the Hessian of J°_t:

$$\nabla^2 J^\circ_t(x_t) = M_t/D_t + \nabla J^\circ_{t+1}\cdot\mathbb{E}_{\xi_t}\left\{\nabla^2_{1,1}f_t\right\}\,. \tag{35}$$

Moreover, $\nabla^2_{1,1} f_t = 0$ since the matrices $A_t(x_t, \xi_t)$ and $b_t(x_t, \xi_t)$ in Assumption 3.1 (vii) are affine in the state x_t, so $\nabla^2 J_t^\circ$ has the simplified expression

$$\nabla^2 J_t^\circ(x_t) = M_t(x_t)/D_t(x_t)\,. \tag{36}$$

Note that the matrix $M_t(x_t)$ is symmetric positive-semidefinite, since it is the sum of the symmetric positive-semidefinite matrices

$$\begin{pmatrix} \mathbb{E}_{\xi_t}\left\{\nabla^2_{1,1} h_t\right\} & \mathbb{E}_{\xi_t}\left\{\nabla^2_{1,2} h_t\right\} \\ \mathbb{E}_{\xi_t}\left\{\nabla^2_{2,1} h_t\right\} & \mathbb{E}_{\xi_t}\left\{\nabla^2_{2,2} h_t\right\} \end{pmatrix} \tag{37}$$

and

$$\begin{pmatrix} \mathbb{E}_{\xi_t}\left\{(\nabla_1 f_t)^T \nabla^2 J_{t+1}^\circ \nabla_1 f_t\right\} & \mathbb{E}_{\xi_t}\left\{(\nabla_1 f_t)^T \nabla^2 J_{t+1}^\circ \nabla_2 f_t\right\} \\ \mathbb{E}_{\xi_t}\left\{(\nabla_2 f_t)^T \nabla^2 J_{t+1}^\circ \nabla_1 f_t\right\} & \mathbb{E}_{\xi_t}\left\{(\nabla_2 f_t)^T \nabla^2 J_{t+1}^\circ \nabla_2 f_t\right\} \end{pmatrix} \tag{38}$$

(the first one is symmetric positive-semidefinite by the convexity and assumed smoothness of h_t, the second one by the convexity and assumed smoothness of J_{t+1}°). So, by formula (31) and Lemma 5.1 one has

$$\inf_{x_t \in X_t} \lambda_{\min}(M_t(x_t)/D_t(x_t)) \geq \alpha_t\,,$$

too. Hence, J_t satisfies (8), and the backward induction argument is proved. $\square$

Proof of Theorem 3.3. We first prove (9). For each $x_{N-1} \in X_{N-1}$, let $H_{N-1}(x_{N-1}, \cdot)$ denote the function

$$\mathbb{E}_{\xi_{N-1}}\left\{h_{N-1}(x_{N-1}, \cdot, \xi_{N-1}) + J_N^\circ(f_{N-1}(x_{N-1}, \cdot, \xi_{N-1}))\right\}\,. \tag{39}$$

It follows by the optimality of g_{N-1}° and formulas (2), (3), and (39), that

$$\begin{aligned} & J_{N-1}^\circ(x_{N-1}) \\ = \; & \tilde{J}_{N-1}(x_{N-1}, g_{N-1}^\circ) \\ = \; & H_{N-1}(x_{N-1}, g_{N-1}^\circ(x_{N-1})) \\ \leq \; & H_{N-1}(x_{N-1}, \tilde{g}_{N-1}^\circ(x_{N-1})) \\ = \; & \tilde{J}_{N-1}(x_{N-1}, \tilde{g}_{N-1}^\circ)\,. \end{aligned} \tag{40}$$

Then, by Taylor's theorem with Lagrange's remainder and the convexity of the set U_{N-1}, one obtains, for some $\bar{u}_{N-1}$ belonging to the line segment between $g_{N-1}^\circ(x_{N-1})$ and $\tilde{g}_{N-1}^\circ(x_{N-1})$,

$$\begin{aligned} & H_{N-1}(x_{N-1}, \tilde{g}_{N-1}^\circ(x_{N-1})) - H_{N-1}(x_{N-1}, g_{N-1}^\circ(x_{N-1})) \\ = \; & \frac{1}{2}(\tilde{a}_{N-1}(x_{N-1}))^T(\nabla^2_{2,2} H_{N-1}(x_{N-1}, \bar{u}_{N-1}))\tilde{a}_{N-1}(x_{N-1})\,, \end{aligned} \tag{41}$$

where for simplicity of notation we have defined $\tilde{a}_{N-1}(x_{N-1}) := \tilde{g}_{N-1}^\circ(x_{N-1}) - g_{N-1}^\circ(x_{N-1})$.

We observe that, by the chain rule and Assumption 3.1 (vii), one has

$$\begin{aligned} & \nabla^2_{u_{N-1},u_{N-1}} J_N^\circ(f_{N-1}(x_{N-1},u_{N-1},\xi_{N-1})) \\ = \; & A^T_{N-1}(x_{N-1},\xi_{N-1}) \nabla^2 J_N^\circ(f_{N-1}(x_{N-1},u_{N-1},\xi_{N-1})) A_{N-1}(x_{N-1},\xi_{N-1})\,. \end{aligned} \tag{42}$$

Then it follows by the definition of $H_{N-1}(x_{N-1},\cdot)$, Theorem 3.2, Assumption 3.1 (v,vii), and formula (42) that one has

$$\inf_{x_{N-1}\in X_{N-1},u_{N-1}\in U_{N-1}} \lambda_{\min}(\nabla^2_{2,2} H_{N-1}(x_{N-1},u_{N-1})) \geq \alpha_{N-1} + \alpha_N \eta_{N-1}\,. \tag{43}$$

This, combined with (41), provides

$$\begin{aligned} & H_{N-1}(x_{N-1},\tilde{g}^\circ_{N-1}(x_{N-1})) - H_{N-1}(x_{N-1},g^\circ_{N-1}(x_{N-1})) \\ \geq \; & \frac{1}{2}\left(\alpha_{N-1} + \alpha_N \eta_{N-1}\right) \|\tilde{a}_{N-1}(x_{N-1})\|^2\,. \end{aligned} \tag{44}$$

Then, (9) is obtained by combining (40) with (44), and taking the supremum norm on X_{N-1}.

We now detail the proof of (10), for $t = N-2,\ldots,0$. In this case, for each $x_t \in X_t$, let $H_t(x_t,\cdot)$ denote the function

$$\mathbb{E}_{\xi_t}\left\{h_t(x_t,\cdot,\xi_t) + J^\circ_{t+1}(f_t(x_t,\cdot,\xi_t))\right\}\,. \tag{45}$$

Hence, one obtains

$$\begin{aligned} & J^\circ_t(x_t) \\ = \; & \tilde{J}_t(x_t, g^\circ_t, g^\circ_{t+1},\ldots,g^\circ_{N-1}) \\ = \; & H_t(x_t, g^\circ_t(x_t)) \\ \leq \; & H_t(x_t, \tilde{g}^\circ_t(x_t)) \\ = \; & \tilde{J}_t(x_t, \tilde{g}^\circ_t, g^\circ_{t+1},\ldots,g^\circ_{N-1}) \\ = \; & \tilde{J}_t(x_t, \tilde{g}^\circ_t, \tilde{g}^\circ_{t+1},\ldots,\tilde{g}^\circ_{N-1}) + \tilde{J}_t(x_t, \tilde{g}^\circ_t, g^\circ_{t+1},\ldots,g^\circ_{N-1}) - \tilde{J}_t(x_t, \tilde{g}^\circ_t, \tilde{g}^\circ_{t+1},\ldots,\tilde{g}^\circ_{N-1}) \\ \leq \; & \tilde{J}_t(x_t, \tilde{g}^\circ_t, \tilde{g}^\circ_{t+1},\ldots,\tilde{g}^\circ_{N-1}) + \|\tilde{J}_{t+1}(\cdot, g^\circ_{t+1},\ldots,g^\circ_{N-1}) - \tilde{J}_{t+1}(\cdot, \tilde{g}^\circ_{t+1},\ldots,\tilde{g}^\circ_{N-1})\|_{\sup(X_{t+1})} \\ \leq \; & \tilde{J}_t(x_t, \tilde{g}^\circ_t, \tilde{g}^\circ_{t+1},\ldots,\tilde{g}^\circ_{N-1}) + \varepsilon_{t+1}\,. \end{aligned} \tag{46}$$

Finally, using (46) and reasoning as above, one obtains

$$\begin{aligned} & \tilde{J}_t(x_t, \tilde{g}^\circ_t, \tilde{g}^\circ_{t+1},\ldots,\tilde{g}^\circ_{N-1}) - J^\circ_t(x_t) + \varepsilon_{t+1} \\ \geq \; & H_t(x_t, \tilde{g}^\circ_t(x_t)) - H_t(x_t, g^\circ_t(x_t)) \\ \geq \; & \frac{1}{2}\left(\alpha_t + \alpha_{t+1}\eta_t\right) \|\tilde{a}_t(x_t)\|^2\,. \end{aligned} \tag{47}$$

where $\tilde{a}_t(x_t) := \tilde{g}^\circ_t(x_t) - g^\circ_t(x_t)$. Then, (10) is obtained from (47) by taking the supremum norm on X_t. □

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